OUTLINE OF A RESTRICTION-CENTERED THEORY OF REASONING AND COMPUTATION IN AN ENVIRONMENT OF UNCERTAINTY AND IMPRECISION

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PART 1
INTRODUCTION

PREAMBLE

- ***The theory which is outlined in this lecture, call it RRC for short, is a departure from traditional approaches to reasoning and computation. A principal advance is an enhanced capability for reasoning and computation in an environment of uncertainty, imprecision and partiality of truth. In large measure, RRC is motivated by the fact that in the real world such environment is the norm rather than exception.***

THE CONCEPT OF A RESTRICTION

- *A concept which has a position of centrality in RRC is that of a restriction. Informally, a restriction is an answer to the question: What is the value of a variable, X? Simple example. I ask: How long will it take me to drive from Berkeley to SF airport? Answers/restrictions:*

a. *1hr and 15min (arithmetic)*
b. *1hr and 15min +/- 15min (interval arithmetic)*
c. *About 1hr and 15min (RRC)*
d. *Usually about 1hr and 15min (RRC)*

CONTINUED

- ***A restriction is a carrier of information about X—information which can assume a variety of forms.***
- ***More concretely, a restriction, R(X), on a variable, X, is a limitation on the values which X can take—a limitation which is induced by what is known or perceived about X. A restriction is singular if the answer to the question is a singleton; otherwise it is nonsingular. Generally,***

PRECISIATED AND UNPRECISIATED RESTRICTIONS

nonsingularity implies uncertainty. A restriction is precisiated if the limitation is mathematically well defined; otherwise it is unprecisiated. Generally, restrictions which are described in a natural language are unprecisiated. To serve as an object of computation, an unprecisiated restriction must be precisiated. A restriction is precisiable if it lends itself to precisiation.

EXAMPLES OF RESTRICTIONS

- ***Restrictions range from very simple to very complex. Examples.***

  $2 \leq X \leq 6$ (possibilistic)

  X is normally distributed with mean m and variants σ^2 (probabilistic)

  X is small (possibilistic)

- ***Usually X is small (possibilistic/probabilistic)***
- ***It is very unlikely that there will be a significant increase in the price of oil in the near future (possibilistic/probabilistic)***

MORE ON RESTRICTIONS

- *The concept of a restriction is significantly more general than the concept of a predicate. In everyday discourse, restrictions are described, for the most part, in natural language. Perceptions are restrictions.*
- *Restrictions play an essential role in human reasoning and cognition.*
- *A natural language may be viewed as a system of restrictions.*
- *To a significant degree, scientific progress is driven by a quest for precisiation of perceptions.*

REPRESENTATION AND COMPUTATION

- *There are two basic issues. First, how can a semantic entity, e.g., a predicate or a proposition, be represented as a precisiated restriction? Simple example. How can the proposition, p: Most Swedes are tall, be represented as a precisiated restriction?*
- *Second, how can restrictions be computed with? An initial treatment of this issue is contained in Zadeh, 1975b.*

EXAMPLE: ADDITION OF RESTRICTIONS

- Usually Robert leaves his office at about 5pm.
- Usually it takes Robert about an hour to get home from work.
- Question: At what time does Robert get home?
- (about 5pm, usually) + (about 1hr, usually)=?
- To answer this question what is needed is the machinery of RRC.

COMPUTATION WITH RESTRICTIONS

- Over the past years, decades and centuries, an enormous resource, call it R, of mathematical constructs, methods and theories has been amassed. The resource, R, has a far-reaching problem-saving capability. But this far-reaching capability falls short of being effective in reasoning and computation with restrictions described in a natural language. There is a basic reason.

PROBLEMS WITH BIVALENT LOGIC

- *In large measure, R is based on the classical, Aristotelian, bivalent logic. Bivalent logic is intolerant of imprecision and partiality of truth. Basically, a natural language is a system for describing perceptions. Perceptions are intrinsically imprecise, reflecting the bounded ability of human sensory organs and ultimately the brain, to resolve detail and store information. Imprecision of perceptions is passed on to natural languages.*

FUZZY LOGIC

- *Imprecision of natural languages is in conflict with precision of bivalent logic. For this reason, bivalent logic is not the right logic for dealing with natural languages. What is needed for this purpose is fuzzy logic, FL.*
- *Informally, FL is a system of reasoning and computation in which the objects of reasoning and computation are classes with unsharp (fuzzy) boundaries.*

THE CONCEPT OF A FUZZY SET

- ***A concept which has a position of centrality in fuzzy logic is that of a fuzzy set. A fuzzy set, A, in a space, U, is a precisiated class, B, which has unsharp (fuzzy) boundaries. Precisiation involves graduation, that is, association of B with a membership function—a function which assigns to each element, u, of U its grade of membership in A, $\mu_A(u)$.***

GRADUATION OF PERCEPTIONS

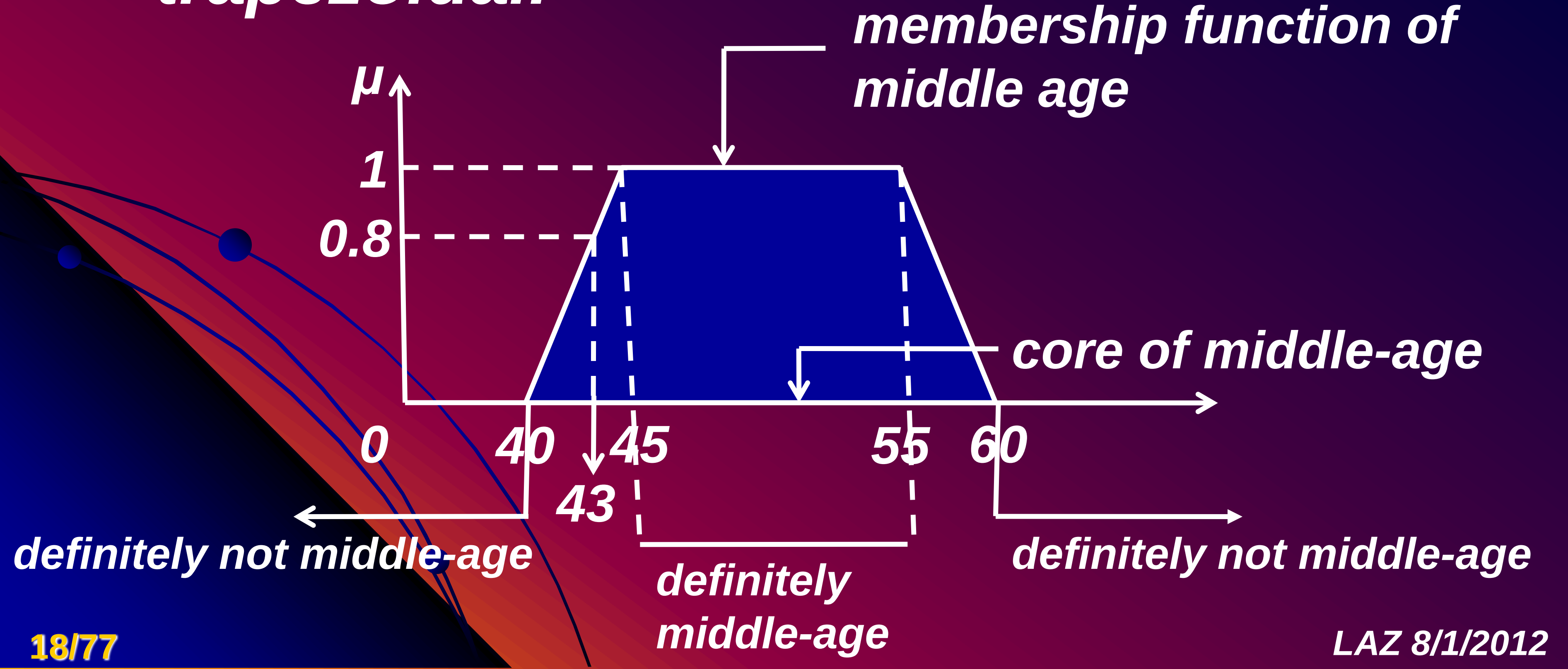
- ***Humans have a remarkable capability to coarsely graduate their subjective perceptions. Humans have no difficulty in answering questions exemplified by: On the scale from 0 to 10, how honest is Robert? On the scale from 0 to 10, how does age 45 fit your perception of middle-age? Other examples: medical questionnaires; grading of term papers.***
- ***This remarkable capability underlies the conceptual structure of fuzzy logic.***

FUZZY LOGIC—EVERYTHING IS A MATTER OF DEGREE

- ***In fuzzy logic everything is, or is allowed to be, a matter of degree, with the understanding that degrees can be fuzzy sets.***
- ***A simple illustration is the following. Consider the proposition, p: Vera is middle-aged, in which middle-aged—a perception of Vera's age—is a class with unsharp boundaries.***

PRECISIATION OF MIDDLE-AGED

- *A simple perception, e.g., middle-aged, may be precisiated as a fuzzy set. For simplicity, the membership function of this fuzzy set may be assumed to be trapezoidal.*



FL-GENERALIZATION

- ***Fuzzy logic provides a basis for what is referred to as FL-generalization. Let T be a bivalent-logic-based theory. FL-generalization of T involves introduction into T of the concept of a fuzzy set, followed by adding to T other concepts and techniques drawn from fuzzy logic, and using them to generalize T.***

FUZZY T

- ***The resulting theory is labeled fuzzy T. Well known examples of FL-generalization are fuzzy control, fuzzy linear programming, fuzzy arithmetic, fuzzy set theory, and fuzzy topology. By now, a number of bivalent logic-based theories have been FL-generalized to some degree. In coming years, more and more bivalent-logic-based theories are likely to be FL-generalized.***

MORE ON FL-GENERALIZATION

- ***FL-generalization applies not only to theories but, more generally, to algorithms, formalisms and concepts.***
- ***Examples. Fuzzy back-propagation algorithm. Fuzzy Markoff algorithm. Fuzzy stability. Fuzzy preference relation.***
- ***FL-generalization opens the door to construction of better models of reality.***

FL-GENERALIZATION IS NEEDED TO CONSTRUCT BETTER MODELS OF REALITY

- In science, it is a deep-seated tradition to employ bivalent logic for definitions of concepts.
- Example. Standard definition of recession is: Decline in GDP for two successive quarters.

  In reality, recession is a matter of degree. Standard definition is patently off based.
- Another example: Grammaticality

RESOURCE, R, AND ENLARGED RESOURCE, R^+

- ***FL-generalized formalisms and theories may be added to resource, R. The result is an augmented resource, R^+. By construction, R^+ has greater generality than R.***

$$\boxed{R} \xrightarrow{\textit{FL-generalization}} \boxed{R^+}$$

PRINCIPAL MODES OF REASONING

- *There are three principal modes of reasoning and computation in RRC.*
- *Type 1 reasoning and computation is R-based. The objects of reasoning and computation are, basically, measurements. The underlying logic is the classical, bivalent logic.*
- *Type 2 reasoning and computation is R^+-based. The objects of reasoning and computation are, basically, measurements and precisiated perceptions.*

TYPE 3 REASONING

- *Type 3 reasoning is the most common form of human reasoning. Almost all of human reasoning in the realms of everyday reasoning, political reasoning and legal reasoning is of Type 3.*

- *In Type 3 reasoning the objects of reasoning are unprecisiated perceptions.*

- *Type 3 reasoning is non-mathematical.*

STRUCTURE OF MODES OF REASONING AND COMPUTATION

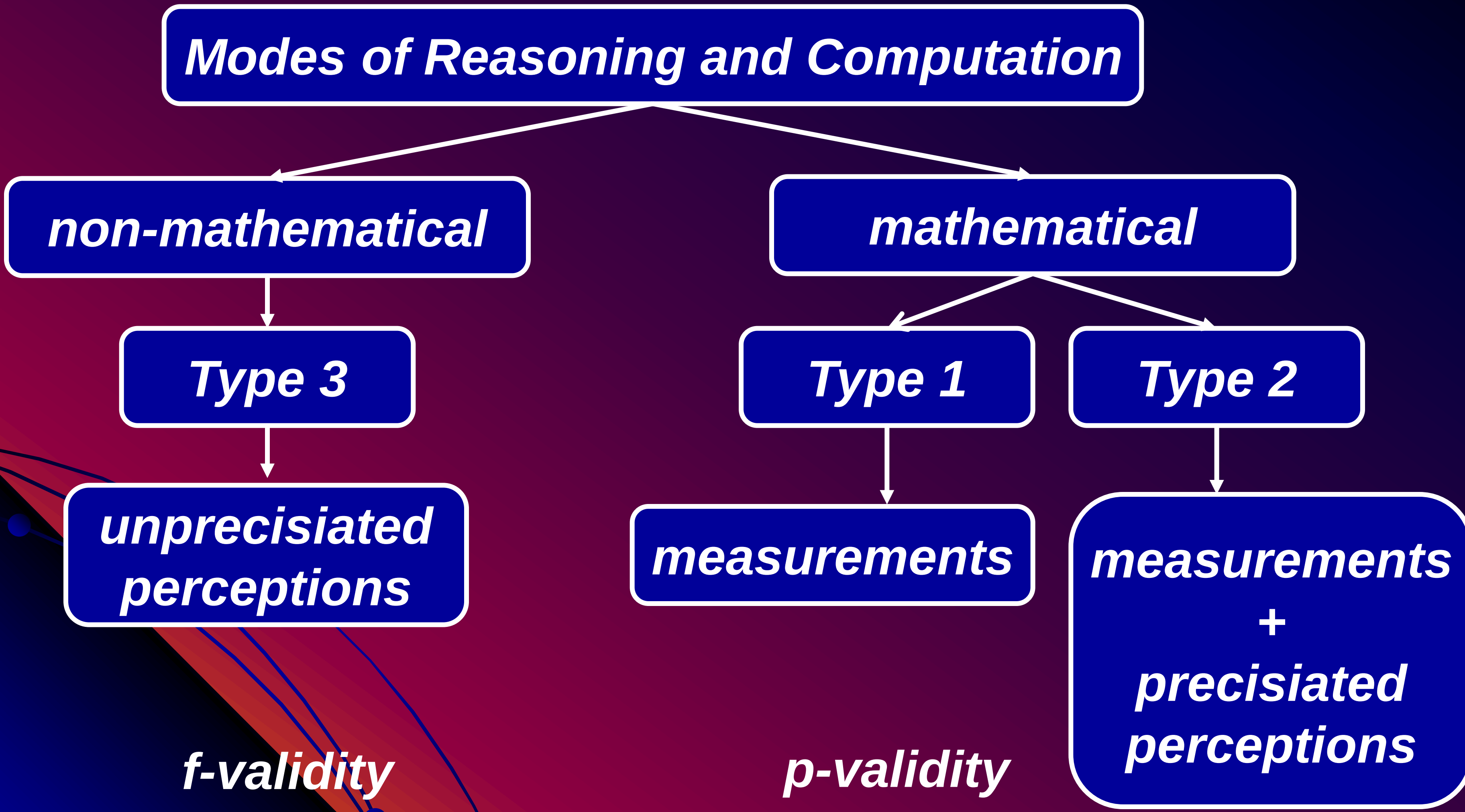


ILLUSTRATION. THE TAXI CAB PROBLEM

- ***I hail a taxi and ask the driver to take me to address A. There are two versions: (a) I ask the driver to take me to A the shortest way; and (b) I ask the driver to take me to A the fastest way. Based on Type 3 reasoning, the driver chooses route a for (a) and route b for (b).***
- ***Assuming that we have a street map, version (a) lends itself to Type 1 reasoning and computation. Version (a) is tractable.***

CONTINUED

- ***Realistic models of version (b) do not lend themselves to Type 1 or Type 2 reasoning, except as an approximation.***
- ***Note. Version (b) lends itself to Type 1 reasoning retrospectively, that is, assuming that traffic history is recorded.***
- ***In version (b), there is a conflict between reality and tractability.***

CONCLUDING REMARK

- ***The concept of a restriction is the centerpiece of RRC. A more detailed discussion of the concept of a restriction is presented in the following section.***

THE CONCEPT OF A RESTRICTION

REPRESENTATION OF A RESTRICTION

- ***The concept of a restriction is closely related to the concept of a generalized constraint (Zadeh 2006)***

- ***The canonical form of a restriction on X, R(X), may be represented as:***

  R(X): X isr R,

  where X is the restricted variable, R is the restricting relation and r is an indexical variable which defines how R restricts X.

HARD AND SOFT RESTRICTIONS

- ***A restriction, R(X), is hard if it is of the form***

$$R(X): X \varepsilon A,$$

where A is a set. A constraint is a hard restriction.

- ***A restriction is soft if it is not hard.***

- ***In RRC, a probabilistic restriction is viewed as a soft restriction.***

DIRECT AND INDIRECT RESTRICTIONS

- ***A restriction on X is direct if it is of the form:***

$$R(X): \quad X \text{ isr } R$$

- ***A restriction on X is indirect if it is of the form:***

$$R(X): \quad f(X) \text{ isr } R,$$

where f is a specified function or functional.

EXAMPLE OF INDIRECT RESTRICTION

$$R(p): \int_R \mu(u)p(u)du \text{ is likely}$$

is an indirect restriction on p.

- ***Note: The term "restriction" is sometimes applied to R.***
- ***Note: Unless stated to the contrary, X is assumed to be real-valued.***

PRECISIATED AND UNPRECISIATED RESTRICTIONS

- *A restriction, X isr R, is precisiated if X, R and r are mathematically well-defined. In particular, if R is a fuzzy set, the restriction is precisiated if the membership function of R is specified.*
- *The restriction is unprecisiated if it is not precisiated. The perceptual meaning, or simply the meaning, of an unprecisiated restriction is the perception which it evokes in one's mind.*

UNPRECISIATED RESTRICTIONS/PERCEPTIONS

- ***Example. If I am told that Vera is middle-aged, without specifying the membership function of middle-aged, my perception of Vera's age is an unprecisiated restriction/ perception. Unprecisiated restrictions/perceptions—described in a natural language—are dominant in human discourse and everyday reasoning.***

CONDITIONAL RESTRICTIONS

- ***A conditional restriction is a restriction which is conditioned on another restriction. A conditional restriction is represented as:***

If X isr R then Y iss S

- ***Example.***
  - ***If pressure is high then volume is low.***
  - ***A fuzzy if-then rule is a conditional restriction.***

PRINCIPAL KINDS OF RESTRICTIONS

- ***The principal restrictions are possibilistic, probabilistic and combinations of possibilistic and probabilistic restrictions.***
- ***In the context of natural languages, restrictions are preponderantly possibilistic. This is why in the case of possibilistic restrictions the value of the indexical variable, r, is simply blank.***

POSSIBILISTIC RESTRICTION

- Possibilistic restriction (r=blank):

$$R(X): \quad X \text{ is } A,$$

where A is a fuzzy set in U with the membership function μ_A. A plays the role of the possibility distribution of X

$$Poss(X=u)= \mu_A(u)$$

EXAMPLES

- *X is small*

restricted variable (X) → *restricting relation (fuzzy set)* (small)

small is the possibility distribution of X

- *Robert is older than Ann*

(Age(Robert), Age(Ann)) is older

restricted variable ((Age(Robert), Age(Ann))) → *restricting relation (fuzzy relation)* (older)

The fuzzy relation "older" is the possibility distribution of ((Age(Robert), Age(Ann))

PROBABILISTIC RESTRICTION

- *Probabilistic restriction (r=p):*

R(X): X isp P,

where P plays the role of the probability distribution of X

$$Prob(u \leq X \leq u+du) = p(u)du,$$

where p is the probability density function of X.

EXAMPLE

- *X is a random variable taking values in a finite set (u_1, …, u_n) with respective probabilities p_1, …, p_n.*

In this case,

- *X isp ($p_1\backslash u_1$+…+$p_n\backslash u_n$),*

in which $p_i\backslash u_i$ means that p_i is the probability of u_i, i=1, …, n.

EXAMPLE

- *X is a normally distributed random variable with mean m and variance σ^2*
- *X isp* $\frac{1}{\;}exp(-(X-m)^2/2\sigma^2)$

restricted variable

restricting relation (probability density function)

Z-RESTRICTION

- *X is a real-valued random variable.*
- *Z-restriction (r=z, s is suppressed) is expressed as*

R(X): X iz Z,

where Z is a combination of possibilistic and probabilistic restrictions defined as

Z: Prob(X is A) is B,

THE CONCEPT OF A Z-NUMBER

in which A and B are fuzzy sets. Usually, A and B are labels drawn from a natural language. The ordered pair, (A,B), is referred to as a Z-number (Zadeh, 2011)

- ***The fuzzy number, B, is a possibilistic restriction on the certainty (probability) that X is A.***

EXAMPLES

- Usually temperature is low $\longrightarrow$ Temperature iz (low, usually)
- Probably John is tall $\longrightarrow$ Height(John) iz (tall, probable).
- Important note:

  Usually X is A,

  where A is a fuzzy set, is a Z-restriction

UNDERLYING PROBABILITY DENSITY FUNCTION

- ***More concretely, B is a possibilistic restriction on the probability of the fuzzy event, X is A. Let p be the probability density function of X, and let μ_A be the membership function of A. The probability of the fuzzy event, X is A, may be expressed as (Zadeh, 1968)***

$$Prob(X \text{ is } A) = \int_R p(u)\mu_A(u)du$$

UNDERLYING PROBABILITY DENSITY FUNCTION

More compactly, Prob(X is A) may be written as the scalar product of p and μ_A

$$Prob(X \text{ is } A) = p \cdot \mu_A,$$

with the understanding that B is an indirect possibilistic restriction on p.

- ***p is referred to as the underlying probability density function of X.***

Z^+-RESTRICTION

- ***Z^+-restriction ($r=z^+$, s is suppressed) is expressed as:***

$$R(X):\quad X\ iz^+\ Z^+,$$

$$Z^+=(Poss(X), Prob(X)),$$

meaning that Z^+ is an ordered pair, (Poss(X), Prob(X)), in which Poss(X) and Prob(X) are, respectively, the possibility and probability distributions of X. Note that Z^+ is more informative than Z.

THE CONCEPT OF A Z^+-NUMBER

More concretely, if $Z=(A,B)$ and p is the underlying probability density function of X, then

$$Z^+=(A, p)$$

- *Correspondingly, If Z is (A,B) and Z^+ is (A,p) then B is an indirect possibilistic restriction on $\mu_A \cdot p$*
- *p is explicit in Z^+ and implicit in Z.*
- *The ordered pair (A,p) is referred to as a Z^+-number.*

EXAMPLE OF Z^+-RESTRICTION

X is the number of eggs which Hans eats for breakfast.

$Poss(X)=1/0+1/1+\ldots+1/6+0.8/7+0.6/8+\ldots$

$Prob(X)=0.3\backslash 0+0.6\backslash 1+0.1\backslash 3$

- *Note that the possibility distribution of X cannot be derived from the probability distribution of X, and vice versa.*

THE CONCEPT OF Z-VALUATION

- ***A Z-valuation is an ordered triple of the form (X,A,B), where X is a real-valued variable and (A,B) is a Z-number. Equivalently, a Z-valuation, (X,A,B), is a Z-restriction on X,***

$$(X,A,B) \longrightarrow X \text{ iz } (A,B).$$

Examples.

- ***(age of Robert, young, very likely)***
- ***(traffic, heavy, usually)***
- ***A Z-valuation may be viewed as a linguistic summary of experience.***

Z-RULES

- ***A Z-rule is a conditional restriction in which the antecedent and consequent are Z-valuations***

$$If(X,A_1,B_1) \text{ then } (Y,A_2,B_2)$$

Example.

- ***If(temperature, low, usually) then (cost of heating, high, usually)***

- ***If(price, high) then (quality, high, usually)***
- ***Z-rules have the potential for playing an important role in representation of possibilistic/probabilistic dependencies.***

Z-INFORMATION

- ***In real-world settings, much of the information in an environment of uncertainty and imprecision may be represented as a collection of Z-valuations and Z-rules—a collection which is referred to as Z-information.***

EXAMPLES OF Z-INFORMATION

- ***Usually Robert leaves his office at about 5pm*** $\longrightarrow$

  (time of departure, about 5pm, usually)

- ***If traffic is heavy, usually travel time is about 1.5 hours*** $\longrightarrow$

  If (traffic, heavy) then (travel time, about 1.5 hours, usually)

Z-INTERPOLATION—A CHALLENGING PROBLEM

- ***If X is A_1 then Y iz (B_1, usually)***

  …

  If X is A_n then Y iz (B_n, usually)

  X is A. What is Y?

This is a generalized version of the problem of interpolation which plays a key role in fuzzy control.

COMPUTATION WITH Z-INFORMATION

- Reasoning and computation with Z-information plays an important role in restriction-centered reasoning and computation, RRC. A very brief exposition of computation in RRC is presented in the following Section. More detail may be found in the Appendix.

RESTRICTION-CENTERED REASONING AND COMPUTATION (RRC)

REPRESENTATION OF INFORMATION AND MEANING AS RESTRICTIONS

- ***There are two postulates which play essential roles in RRC. First, the information postulate, IP. Second, the meaning postulate, MP.***
- ***The information postulate, IP, equates information to a restriction***

information=restriction

What IP implies is that information about the value of a variable is conveyed by restricting the values which the variable can take.

MEANING POSTULATE

- ***This interpretation of information is considerably more general than the entropy-based definition of information in information theory.***

- ***The meaning postulate, MP, equates meaning to restriction***

meaning=restriction

MORE ON MEANING POSTULATE

- *What MP implies is that the meaning of a proposition, p, with p viewed as a carrier of information about a variable, X, may be represented as a restriction on the values which X can take. In symbols,*

$$p \longrightarrow X \text{ isr } R$$

- *X is explicit or implicit in p.*

EXAMPLES

- *p: Robert is young ⟶ Age(Robert) is young*

  ↑ X *↑ R*

- *p: Most Swedes are tall ⟶*

  Proportion(tall Swedes/Swedes) is most

  ↑ X *↑ R*

- *p: Usually it takes Robert about an hour to get home from work ⟶ Travel time from office to home iz (approximately 1 hr, usually)*

PRECISIATION OF MEANING

- ***Representation of the meaning of p as a restriction is referred to as precisiation of meaning of p or, more simply, precisiation of p. The same applies to words, phrases, questions, commands and other types of semantic entities.***
- ***Informally, the subjective meaning of a semantic entity is the perception which it evokes in one's mind.***
- ***Precisiation of propositions drawn from a natural language falls within the province of Computing with Words (CWW) (Zadeh, 2012)***

COMPUTING WITH WORDS (CWW)

- ***Precisiation of meaning is a preliminary to computation with information described in natural language. Computation with information described in natural language has a position of centrality in Computing with Words (CWW). In the following, attention will be focused on computation with Z-information, and, more particularly, computation with Z-numbers. Because of limitations on time, our discussion is compressed.***

THE EXTENSION PRINCIPLE

- ***A mathematical formalism which plays an essential role in computation with restrictions is the extension principle (Zadeh, 1965, 1975a I II III) A brief discussion of the extension principle is presented in the following Section.***

THE EXTENSION PRINCIPLE

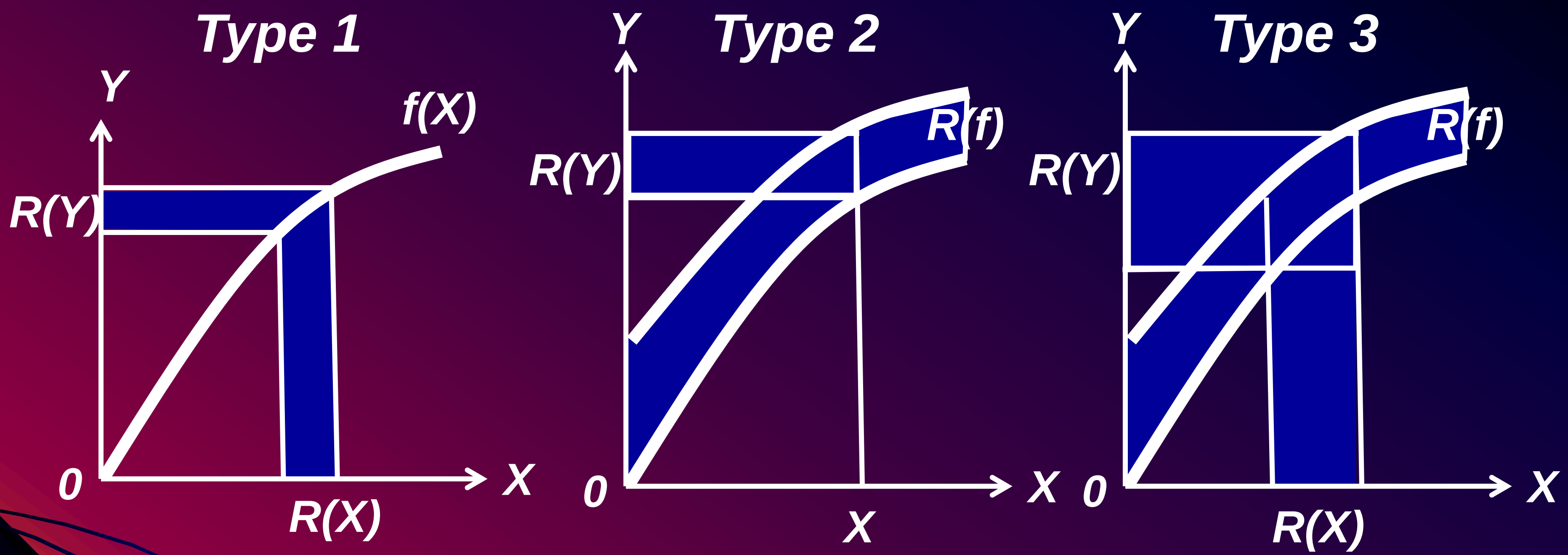
THE EXTENSION PRINCIPLE

- ***The extension principle is not a single principle. The extension principle is a collection of computational rules in which the objects of computation are various types of restrictions. More concretely, assume that Y is a function of X, Y=f(X), where X may be an n-ary variable. Assume that what we have is imperfect information about X and/or f, implying that what we know are restrictions on X and/or f, respectively, R(X) and R(f).***

MORE ON THE EXTENSION PRINCIPLE

These restrictions induce a restriction on Y, R(Y). The extension principle relates to computation of R(Y) given R(X) and R(f). There are three types of the extension principle. In Type 1, we know f and R(f). In Type 2, we know X and R(f). In Type 3, we know R(X) and R(f). There are different versions of each type depending on the nature of restrictions. In what follows, because of limitations of time, we consider only two basic versions of Type 1. Other versions are discussed in the Appendix.

TYPES OF THE EXTENSION PRINCIPLE



Type 1	Type 2	Type 3
Y=f(X)	Y=f(X)	Y=f(X)
R(X)	R(f)	R(X)
		R(f)
? R(Y)	? R(Y)	? R(Y)

- *Note. R(f) is usually described as a collection of fuzzy if-then rules (Zadeh 1976)*

POSSIBILISTIC EXTENSION PRINCIPLE

- *The simplest version (Zadeh 1965) is one in which the restriction is possibilistic and direct. This version of the extension principle is expressed as:*

CONTINUED

$$\frac{\begin{array}{c} Y=f(X) \\ R(X): X \text{ is } A \end{array}}{R(Y)(f(A)): \mu_Y(v)=\sup_u(\mu_A(u))}$$

subject to

$$v=f(u),$$

where μ_A and μ_Y are the membership functions of A and Y, respectively

A MORE GENERAL VERSION

- ***A slightly more general version (Zadeh 1975a) is one in which R(X) is possibilistic and indirect.***

$$\frac{\begin{gathered} Y=f(X) \\ R(X):\ g(X) \text{ is } A \end{gathered}}{R(Y)(f(A)):\ \mu_Y(v)=\sup_u(\mu_A(g(u)))}$$

subject to

$$v=f(g(u))$$

A SIMPLE EXAMPLE

- *Given, p: Most Swedes are tall.*
- *Question, q: What is the average height of Swedes?*
- *The first step involves precisiation of p and q. For this purpose, it is expedient to employ the concept of a height density function, h.*
- *h(u)du=proportion of Swedes whose height lies in the interval [u, u+du]. If h_{min} and h_{max} are, respectively, the minimum and maximum heights in the population, we have:*

$$\int_{h_{min}}^{h_{max}} h(u)du=1$$

CONTINUED

- ***In terms of the height density function, precisiations of q and p may be expressed as q* and p*:***

$$q*: ?have = \int_{h_{min}}^{h_{max}} u h(u) du$$

$$p*: \int_{h_{min}}^{h_{max}} \mu_{tall}(u) h(u) du \text{ is most,}$$

where μ_{tall} is the membership function of tall.

CONTINUED

- Applying the basic, indirect, possibilistic version of the extension principle, computation of h_{ave} is reduced to the solution of the variational problem

$$\mu_{h_{ave}}(v) = \sup_h \mu_{most}\left(\int_{h_{min}}^{h_{max}} \mu_{tall}(u)h(u)du\right)$$

subject to $v = \int_{h_{min}}^{h_{max}} u h(u) du$

and $\int_{h_{min}}^{h_{max}} h(u) du = 1$

IMPORTANT NOTE

- It is important to note that the solution is a fuzzy set which is a restriction on the values which h_{ave} can take. The fuzzy set may be viewed as the set of all values of h_{ave} which are consistent with the given information, p, with the understanding that consistency is a matter of degree.
- Note that the solution involves reasoning and computation of Type 2.

CONCLUDING REMARK

- *For purposes of reasoning and computation in RRC, what are needed—in addition to possibilistic versions of the extension principle—are versions in which restrictions are probabilistic, Z^+-restrictions and Z-restrictions. Because of restrictions on time, these versions together with examples, and quasi-formalization of Type 3 reasoning, are discussed in Part 2 (Appendix).*

PART 2 (APPENDIX)

MORE ON THE EXTENSION PRINCIPLE AND APPLICATIONS

BINARY VERSION OF BASIC EXTENSION PRINCIPLE

- If $X=(X_1,X_2)$ and X_1 and X_2 are independently restricted, then the extension principle reads

$$Y=f(X_1,X_2)$$

$$R(X_1): X_1 \text{ is } A_1$$

$$\underline{R(X_2): X_2 \text{ is } A_2}$$

$$R(Y)(f(A_1,A_2)): \mu_Y(v)=\sup_{u_1,u_2}(\mu_{A_1}(u_1)\wedge\mu_{A_2}(u_2))$$

subject to $v=f(u_1,u_2)$,

where $\wedge$ is conjunction

PROBABILISTIC EXTENSION PRINCIPLE

- For simplicity, it will be assumed that X takes values in a finite set (u_1, …., u_n), with respective probabilities p_1, …, p_n. In this case, the extension principle reads

$$Y=f(X)$$

$$\frac{R(X)\text{: } X \text{ isp } p_1\backslash u_1+\cdots+p_n\backslash u_n}{R(Y)(f(p))\text{: } Y \text{ isp } q_1\backslash v_1+\cdots+q_m\backslash v_m},$$

where $q_j=\sum_1^n p_i$, subject to $v_j=f(p_i)$

Continuous versions may be found in textbooks on probability theory. Continuous versions are more complex.

PROBABILISTIC EXTENSION PRINCIPLE WITH A POSSIBILISTIC RESTRICTION

$$\frac{\begin{array}{c} Y=f(p) \\ R(p):\ g(p) \text{ is } A \end{array}}{R(Y)(f(A)):\ \mu_Y(q)=\sup_p \mu_A(g(p))}$$

subject to

$$q=f(p).$$

p is a probability density function in R.

A is a fuzzy set in the space of probability density functions.

Z^+-EXTENSION PRINCIPLE

- *In this case, the restriction on X is a Z^+-restriction.*

$$\frac{\begin{array}{c} Y=f(X) \\ R(X): X\ iz^+\ (A,p) \end{array}}{R(Y)(f(A,p)): R(Y)\ iz^+\ (f(A), f(p)),}$$

where A is a fuzzy set which defines the possibility distribution of X, and p is the underlying probability density function of X.

Z-EXTENSION PRINCIPLE

- *In this case, the restriction on X is a Z-number.*

$$\frac{\begin{gathered} Y=f(X) \\ R(X)\text{: } X \text{ iz } (A,B) \end{gathered}}{R(Y)(f(A,B))\text{: } Y \text{ iz } (f(A),\, C)},$$

where C is the certainty of f(A). Computation of C is fairly complex; it involves an application of the Z^+-extension principle, followed by an application of the probabilistic extension principle with a probabilistic restriction, p·A is B. As an illustration, computation of the sum of two Z-numbers is described in the Appendix.

COMPUTATION WITH Z-NUMBERS

- ***Computation with Z-numbers is an important area within computation with Z-restrictions. Computation with Z-numbers requires the use of the Z-version of the extension principle. An example which involves computation of the sum of two Z-numbers is discussed in the following.***

COMPUTATION OF THE SUM OF Z-NUMBERS

- Let $X=(A_X,B_X)$ and $Y=(A_Y,B_Y)$. The sum of X and Y is a Z-number, $Z=(A_Z,B_Z)$. The sum of (A_X,B_X) and (A_Y,B_Y) is defined as:

$$(A_X,B_X) + (A_Y,B_Y)= (A_X+A_Y,B_Z),$$

where A_X+A_Y is the sum of fuzzy numbers A_X and A_Y computed through the use of fuzzy arithmetic. The main problem is computation of B_Z.

CONTINUED

- ***Let p_X and p_Y be the underlying probability density functions in the Z-valuations (X,A_X,B_X) and (Y,A_Y,B_Y), respectively. If p_X and p_Y where known, the underlying probability density function in Z would be the convolution of p_X and p_Y, $p_Z = p_X \circ p_Y$, expressed as:***

$$p_{X+Y}(v) = \int_R p_X(u)p_Y(v-u)du$$

where R is the real line

CONTINUED

- *What we know are not p_X and p_Y, but restrictions on p_X and p_Y which are expressed as:*

$$\int_R \mu_{A_X}(u) p_X(u) du \text{ is } B_X$$

$$\int_R \mu_{A_Y}(u) p_Y(u) du \text{ is } B_Y$$

CONTINUED

- *Using the extension principle we can compute the restriction on p_Z. It reads:*

$$\mu_{p_Z}(p_Z)=\sup_{p_X,p_Y}(\mu_{B_X}(\int_R \mu_{A_X}(u)p_X(u)du)\wedge$$

$$\mu_{B_Y}(\int_R \mu_{A_Y}(u)p_Y(u)du))$$

subject to $p_Z=p_X\circ p_Y$

$$\int_R p_X(u)du=1, \int_R p_Y(u)du=1$$

CONTINUED

- If p_Z were known, B_Z would be given by:

$$B_z = \int_R \mu_{A_Z}(u) p_Z(u) du,$$

where

$$\mu_{A_Z}(u) = \sup_v (\mu_{AX}(v) \wedge \mu_{AY}(u-v))$$

Since what we know at this point is a restriction on p_Z, it is necessary to employ the extension principle to compute the restriction on B_Z. The result may be expressed as:

CONTINUED

- $\mu_{B_Z}(w) = \sup_{p_Z} (\mu_{p_Z}(p_Z))$

subject to

$$w = \int_R \mu_{A_Z}(u) p_Z(u) du,$$

where $\mu_{p_Z}(p_Z)$ was derived earlier. In principle, this completes computation of the sum of Z-numbers, Z_X and Z_Y.

NOTE

- ***Computation with Z-numbers is a move into a largely unexplored territory. A variety of issues remain to be explored. One such issue is that of informativeness of results of computations. A discussion of this issue may be found in Zadeh 2011. To enhance informativeness and reduce complexity of computations, it may be expedient to make simplifying assumptions about the underlying probability distributions.***

QUASI-FORMALIZATION OF UNPRECISIATED REASONING (TYPE 3) f-GEOMETRY

PREAMBLE

- ***In this Section, we briefly address a basic question. Can unprecisiated, Type 3 reasoning be formalized, and if so, how? The question is in need of clarification.***
- ***Historically, a constantly growing number of areas within the world of humanities and human-centered systems, call it world A, for short—areas in which little or no mathematics was employed, have metamorphosed into mathematically sophisticated theories with wide-ranging practical applications.***

CONTINUED

- *Familiar examples are economics, decision analysis and theories of natural language. In this metamorphosis, unprecisiated perceptions were precisiated, opening the door to reasoning and computation of Type 1 and Type 2. This process is not what is meant by formalization of Type 3 reasoning.*

CONTINUED

- Employment of reasoning of Type 1 and Type 2 in world A has its limitations. Little or no success has been achieved in dealing with problems in areas such as theories of fairness, legal reasoning, political debates, human discourse, etc. What is widely unrecognized is that in realistic settings such areas are intrinsically unsuited for formalization within the conceptual structure of traditional mathematics.

CONTINUED

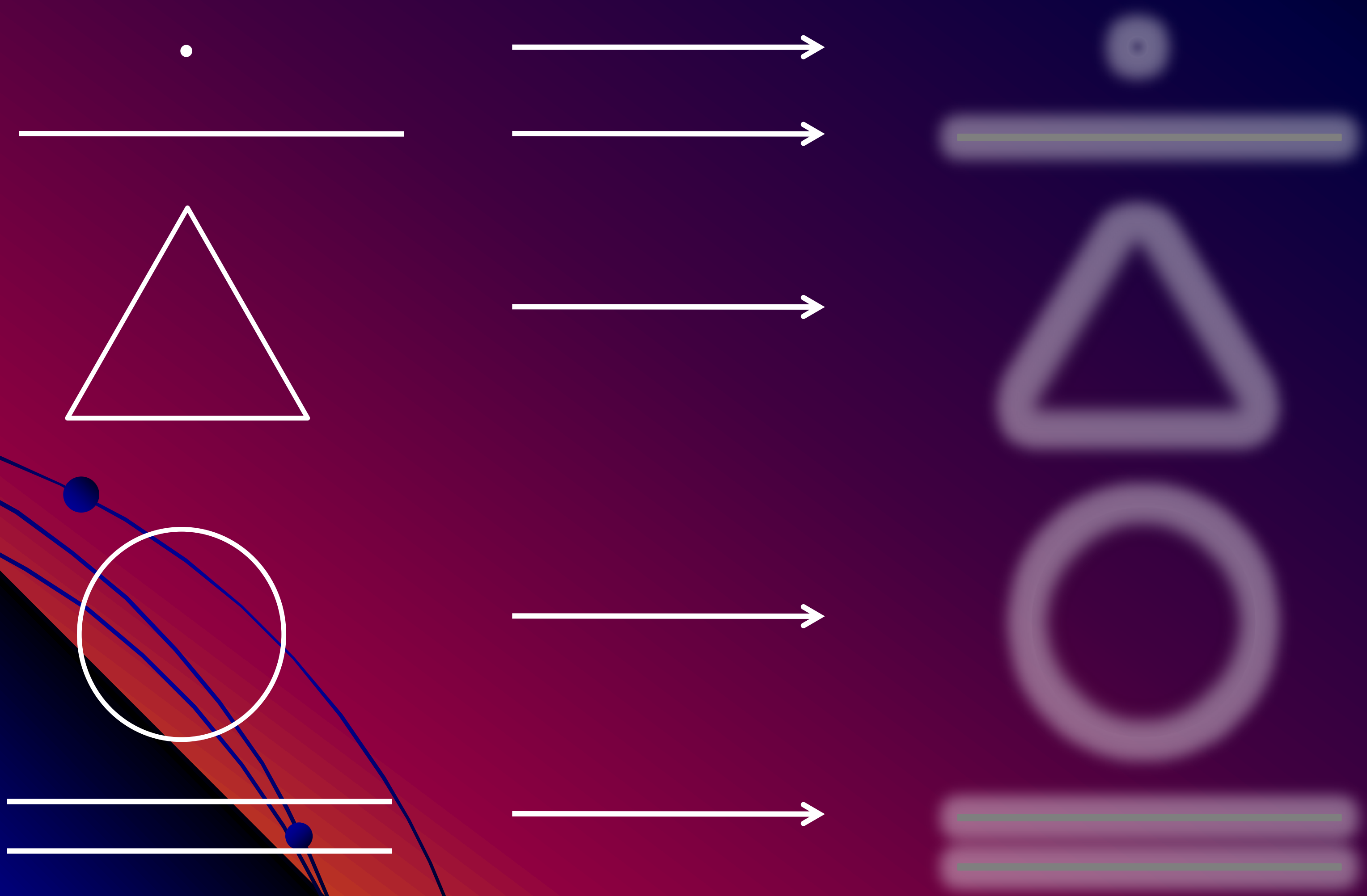
- ***What is suggested in the following is that it may be necessary to settle for what may be called quasi-formalization—a mode of formalization which lies outside the boundaries of traditional mathematics. An informative example of quasi-formalization is what may be called f-geometry. In f-geometry, there are no formal concepts, no formal definitions and no formal theorems. A brief description of f-geometry is presented in the following.***

f-VALID REASONING AND f-GEOMETRY

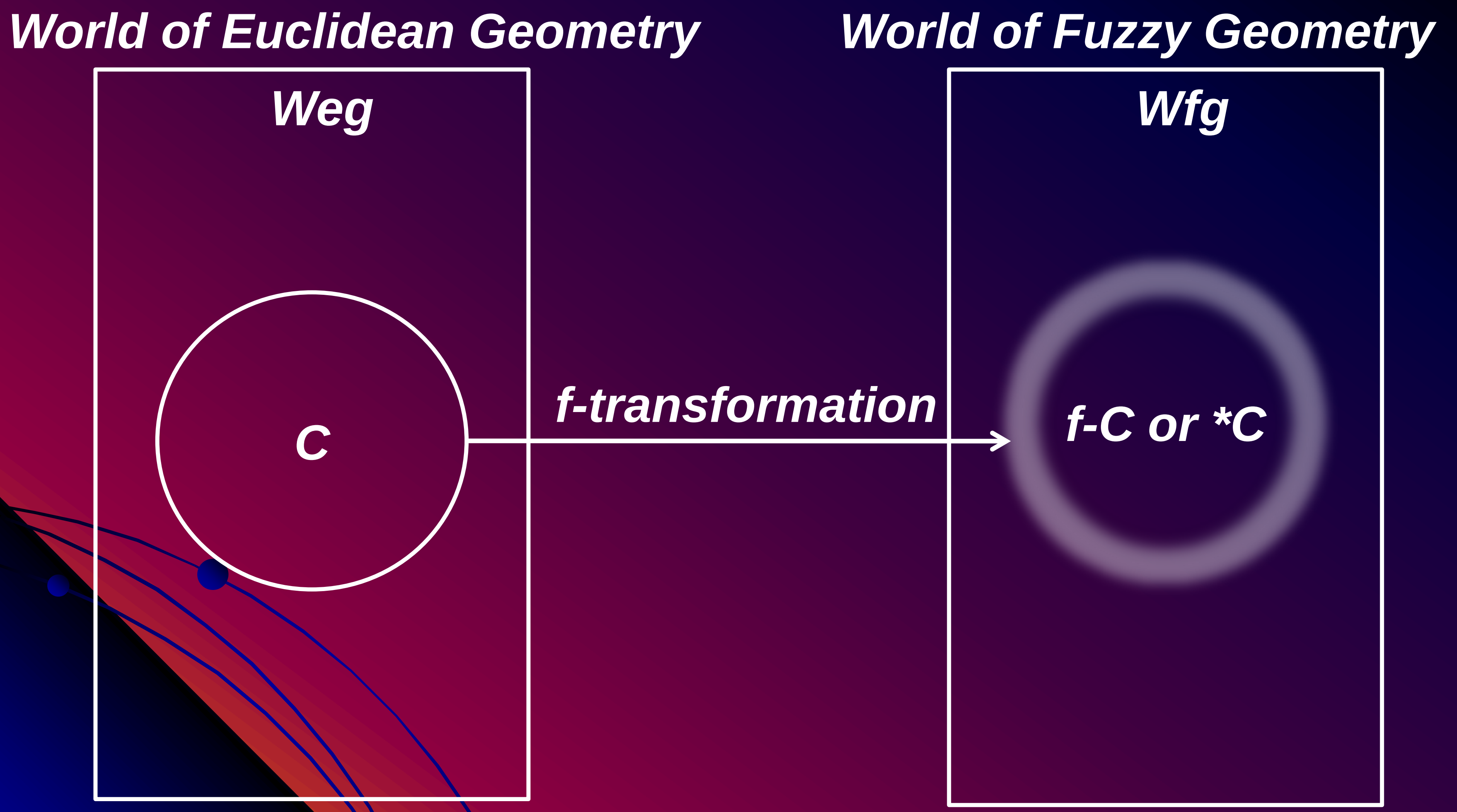
- ***In f-geometry the drawing instrument is a spray pen with an adjustable spray pattern. Drawing is done by hand.***
- ***f-geometry is unrelated to Poston's fuzzy geometry (Poston, 1971), coarse geometry (Roe, 1996), fuzzy geometry of Rosenfeld (Rosenfeld, 1998), fuzzy geometry of Buckley and Eslami (Buckley and Eslami, 1997), fuzzy geometry of Mayburov (Mayburov, 2008), and fuzzy geometry of Tzafestas (Tzafestas et al, 2006).***

f-TRANSFORMATION

Informally, in the context of f-geometry, an f-transform of C is the result of execution of the instruction: Draw C by hand with a spray pen.



f-TRANSFORMATION AND f-GEOMETRY



Note that fuzzy figures, as shown, are not hand drawn. They should be visualized as hand drawn figures.

f-CONCEPTS IN f-GEOMETRY

- f-point
- f-line
- f-triangle
- f-parallel
- f-similar
- f-circle
- f-median
- f-perpendicular
- f-bisector
- f-altitude
- f-concurrence
- f-tangent
- f-definition
- f-theorem
- f-proof
- …

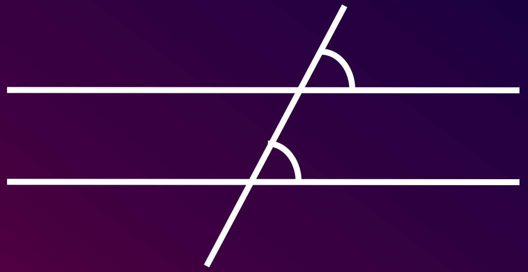
f-TRANSFORMATION

- ***The cointension of f-C is a qualitative measure of the proximity of f-C to its prototype, C. A fuzzy transform, f-C, is cointensive if its proximity to C is close. Unless stated to the contrary, f-transforms are assumed to be cointensive.***

- ***A key idea in f-geometry is the following: if C is p-valid (provably valid) then its f-transform, f-C, is f-valid (fuzzily valid) with a high validity index. As a simple example, consider the definition, D, of parallelism in Euclidean geometry.***

f-TRANSFORMATION OF DEFINITIONS

D: Two lines are parallel if for any transversal that cuts the lines the corresponding angles are congruent.

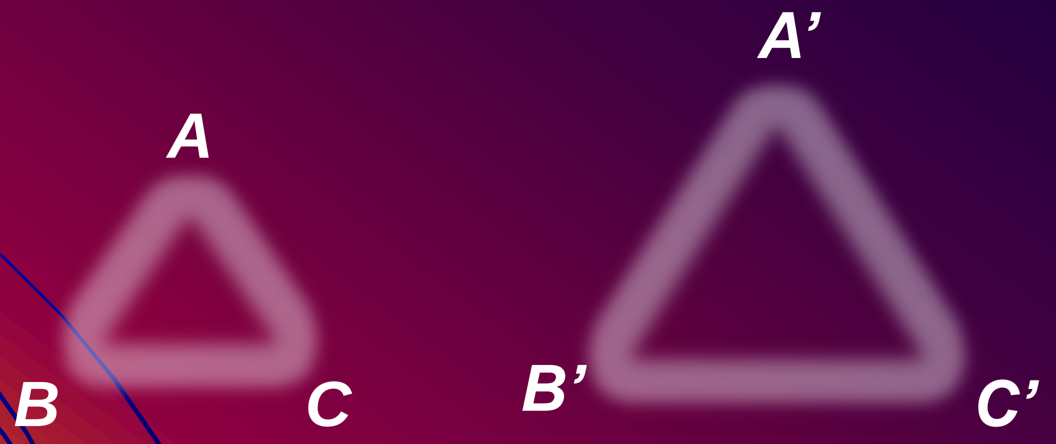


- ***f-transform of this definition reads:***

f-D: Two f-lines are f-parallel if for any f-transversal that cuts the lines the corresponding f-angles are f-congruent.

f-TRANSFORMATION OF DEFINITIONS

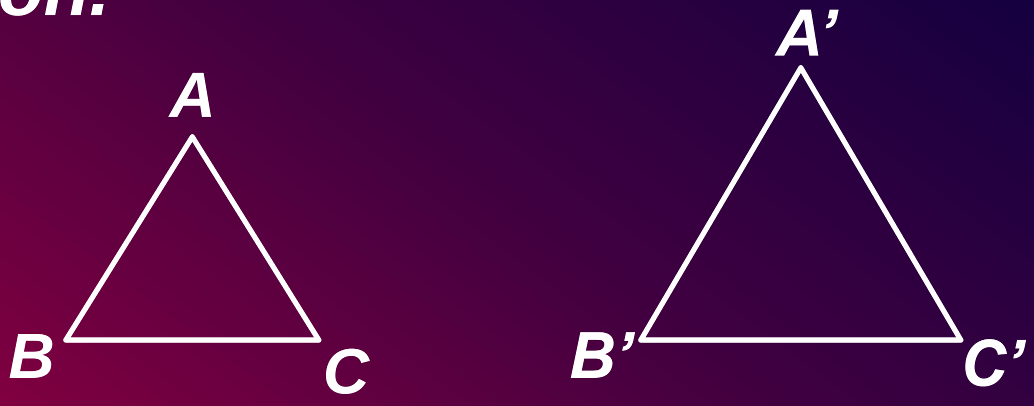
- ***In Euclidean geometry, two triangles are similar if the corresponding angles are congruent. Correspondingly, in f-geometry two f-triangles are f-similar if the corresponding f-angles are f-congruent.***



f-TRANSFORMATION OF PROPERTIES

Simple example

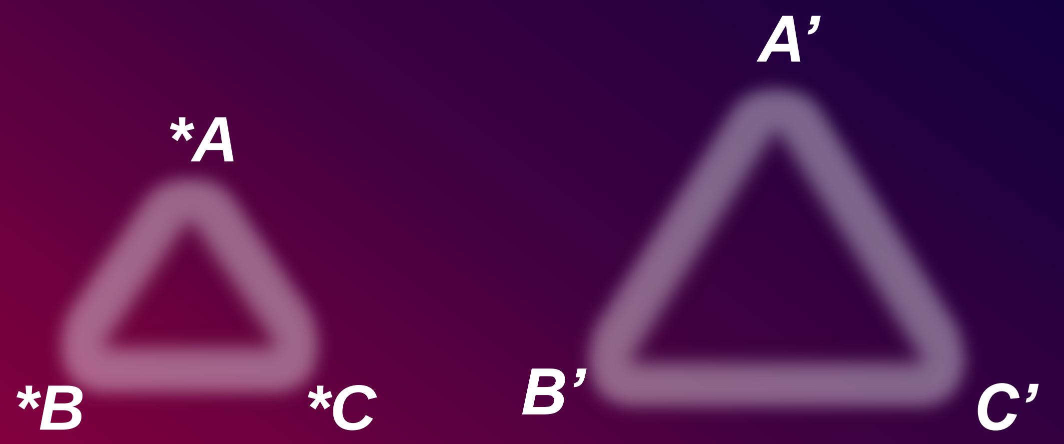
- ***P: if the triangles A, B, C and A’, B’, C’ are similar, then the corresponding sides are in proportion.***



$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$$

f-TRANSFORMATION OF PROPERTIES

- ***P: if the f-triangles *A, *B, *C and A’, B’, C’ are f-similar, then the corresponding sides are in f-proportion.***



$$\frac{{*A}{*B}}{A'B'} = \frac{{*B}{*C}}{B'C'} = \frac{{*C}{*A}}{C'A'}$$

f-TRANSFORMATION OF THEOREMS

- ***An f-theorem in f-geometry is an f-transform of a theorem in Euclidean geometry.***

Simple example

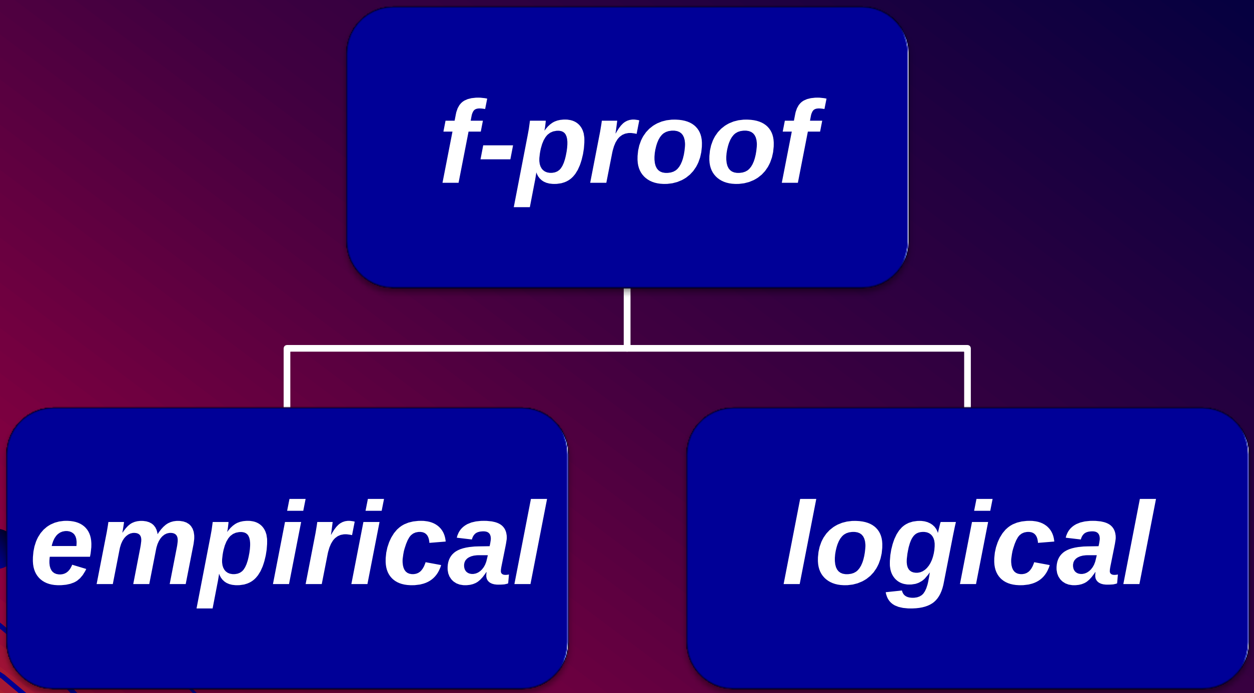
- ***an elementary theorem, T, in Euclidean geometry is:***

T: the medians of a triangle are concurrent.

- ***A corresponding theorem, f-T, in f-geometry is:***

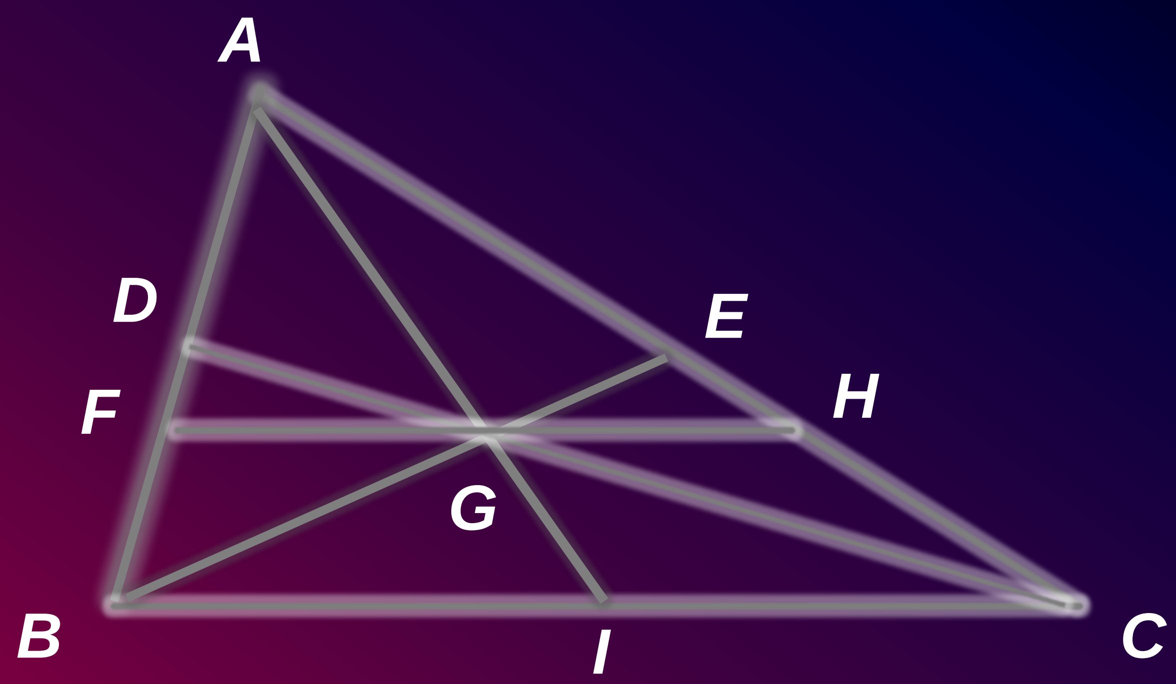
f-T: the f-medians of an f-triangle are f-concurrent.

THE CONCEPT OF f-PROOF



A logical f-proof is an f-transform of a proof in Euclidean Geometry.

LOGICAL f-PROOF—A SIMPLE EXAMPLE



D, E are f-midpoints
DE is f-parallel to BC
FH is f-parallel to BC
AGI is an f-line passing through f-point G
f-triangles EGH and EBC are f-similar

f-triangles DFG and DBC are f-similar
f-proportionality of corresponding sides of f-triangles implies that G is f-midpoint of FH
G is f-midpoint of FH implies that I is f-midpoint of BC
I is f-midpoint of BC implies that the f-medians are f-concurrent

A KEY OBSERVATION

- *The f-theorem and its f-proof are f-transforms of their counterparts in Euclidean geometry. But what is important to note is that the f-theorem and its f-proof could be arrived at without any reference to their counterparts in Euclidean geometry.*

A KEY OBSERVATION

- ***This suggests an intriguing possibility of constructing, in various fields, independently arrived at systems of f-concepts, f-definitions, f-theorems, f-proofs and, more generally, f-reasoning and f-computation. In the conceptual world of such systems, p-validity has no place.***

f-GEOMETRY AND BEYOND

- ***In summary, f-geometry may be viewed as the result of application of f-transformation to Euclidean geometry.***

- ***The concept of f-transformation has a potential for application in fields other than Euclidean geometry. An important step in this direction is taken in the work of R. Aliev et al, 2011 on decision-analysis.***

CONCLUDING REMARK

- ***The theory outlined in this lecture, RRC, enhances our capability for reasoning and computation in an environment of uncertainty, imprecision and partiality of truth.***
- ***RRC may be viewed as a first step in this direction.***

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